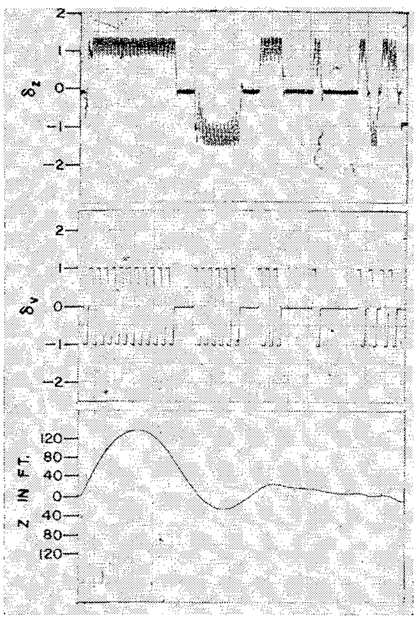

Fig. 5 Record of bang-bang control flight.

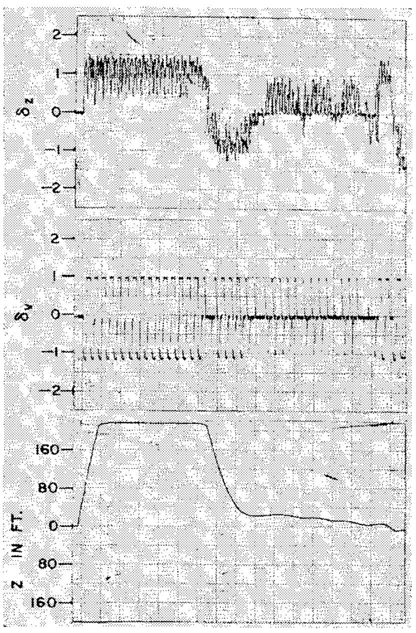

Fig. 6 Record of proportional control flight.

made from a simulated flight using a bang-bang type of control system, in which the length of time that control surfaces are deflected is constant. For this particular flight, the control surface deflection time was made equal to the time the missile requires to roll 150°. The recordings of Fig. 6 were made from a simulated flight using a proportional type of control system in which the length of time the control surfaces are deflected is proportional to the magnitude of the target error angle. The δ_V recordings are made with respect to the rolling missile axis system and are control surface deflections plotted as a function of time. The δ_Z and Z plots are made with reference to the space-stabilized axis system. The δ_Z is the effective control surface deflection in the pitch plane plotted as a function of time, and Z is the plot of missile trajectory.

In conclusion, the single-plane simulation of a rolling missile has made possible the optimization of a large number of system characteristics, and the simplicity of the simulation makes it easy to set up, check out, and make changes.

Explicit Rendezvous Guidance Mechanization

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THIS paper presents a set of guidance equations based on an exact solution of the equations of motion of the rendezvous vehicle. The rendezvous is accomplished in two thrusting periods: injection into the rendezvous trajectory, and nulling the closing velocity. By selecting the time of rendezvous, a free-fall trajectory may be determined which starts at the present position of the vehicle and passes through the rendezvous point at the designated time. The guidance system determines the velocity required to achieve this free-fall trajectory, and drives the difference between the required and the actual velocity to zero. When the required velocity is achieved, the engines are cut off, and the vehicle coasts to rendezvous. Near the rendezvous point the engines are reignited to null the closing velocity. The terminal phase of rendezvous is already discussed adequately in the literature.[1, 2] Therefore, this note is concerned mainly with determining the velocity vector necessary to achieve the free-fall trajectory that passes between two points in space in a given time increment.

Gauss' method of orbit determination[3] could be used, but the equations involve numerous functions, which require too much time to solve in updating the required velocity. These time lags affect the accuracy of the updating, the steering loop, and the mechanics of cutoff. To surmount this problem, the mechanization equations discussed in this paper use dual major and minor cycle.

Mechanization Equations

The use of a fixed time of rendezvous means that this trajectory must pass through a given point in space at the rendezvous time. The total time rate of change of the required velocity $\bar{V}_R$ for this maneuver is

$$\frac{d\bar{V}_R}{dt} = \frac{\partial \bar{V}_R}{\partial t} + \left[\frac{\partial \bar{V}_R}{\partial \bar{P}}\right]\frac{d\bar{P}}{dt} \tag{1}$$

where $[\partial \bar{V}_R/\partial \bar{P}]$ is a three-by-three matrix of partial derivatives, $\bar{P}$ is the present position, and $d\bar{P}/dt$ the present velocity $\bar{V}$ of the vehicle. This equation can be put in a slightly different form by noting that, if the vehicle were on the rendezvous trajectory, its velocity would be $\bar{V}_R$ and that the total time rate of change of the required velocity would be the gravitational acceleration:

$$\bar{g} = (\partial \bar{V}_R/\partial t) + [\partial \bar{V}_R/\partial \bar{P}]\bar{V}_R \tag{2}$$

Vehicle steering and engine cutoff will probably be based upon the velocity-to-be-gained $\bar{V}_g$:

$$\bar{V}_g = \bar{V}_R - \bar{V} \tag{3}$$

Noting that the vehicle acceleration can be written as

$$d\bar{V}/dt = \bar{g} + \bar{A}_T \tag{4}$$

where $\bar{A}_T$ is acceleration because of thrust and other nongravitational forces, and combining Eqs. (1–4) yields (see Ref. 4)

$$d\bar{V}_g/dt = -{}_T\bar{A} - [\partial \bar{V}_R/\partial \bar{P}]\bar{V}_g \tag{5}$$

An explicit mechanization of Eq. (5) would require knowing the partial derivative matrix $[\partial \bar{V}_R/\partial \bar{P}]$ for all possible

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Table 1 Guidance accuracy of mechanization

Time between $\bar{V}_g$ corrections, sec	Terminal acceleration, ft/sec²	Rendezvous miss, ft
14	15	300
7	15	200
7	30	250
7	60	750
3	60	200

positions and transit times and would result in excessive computer storage requirements.

Therefore, a dual mode of computer operation is used. A minor cycle operation updates $\bar{V}_g$ by Eq. (5) using an approximate value of $[\partial\bar{V}_R/\partial\bar{P}]$; it can be performed rapidly enough to provide for smooth steering signals and an accurate engine cutoff. The major cycle operation then corrects for the drift resulting from using Eq. (5) and updates the partial derivatives. It can be performed slowly, allowing the computer sufficient time to solve the two-body equations of motion.

The technique for correcting the drift of $\bar{V}_g$ is begun by using $\bar{P}$ and the current estimate of the required velocity $\bar{V}_R'$ to find the resulting position $\bar{P}(T)$ at the rendezvous time. The rendezvous miss $\bar{M}$ caused by the error in $\bar{V}_R'$ is

$$\bar{M} = \bar{R} - \bar{P}(T) \tag{6}$$

where $\bar{R}$ is the nominal rendezvous position. If the estimated required velocity is perturbed by an amount $\Delta\bar{V}_R$ to correct for the miss, then,

$$\begin{aligned} \bar{R} &= \bar{P}(T) + [\partial\bar{P}(T)/\partial\bar{V}_R']\Delta\bar{V}_R \\ \Delta\bar{V}_R &= [\partial\bar{P}(T)/\partial\bar{V}_R']^{-1}\bar{M} \end{aligned} \tag{7}$$

It can be shown that consideration of the combined effect of $\Delta\bar{V}_R$ and $\Delta\bar{P}$ upon $\bar{P}(T)$ leads to the following equation:

$$\left[\frac{\partial\bar{V}_R}{\partial\bar{P}}\right] = -\left[\frac{\partial\bar{P}(T)}{\partial\bar{V}_R'}\right]^{-1}\left[\frac{\partial\bar{P}(T)}{\partial\bar{P}}\right] \tag{8}$$

The partial derivative matrices on the right-hand side of Eqs. (7) and (8) can be found by numerical differentiation with the onboard computer. The differentiation is performed by determining the effect upon $\bar{P}(T)$ of a small perturbation in the initial position of the vehicle or in the estimate of the required velocity. Dividing the change in $\bar{P}(T)$ by the perturbation gives an estimate of the partial derivative.

The foregoing discussion is concerned with updating $\bar{V}_g$, but it can be seen that the technique is stable and will converge upon the correct value even if the initial estimate is in error. The question of choosing the nominal engine ignition time and nominal rendezvous time are left open. Analysis based upon impulsive corrections and two-body mechanics can be used to determine these quantities in order to minimize fuel consumption, error sensitivities, etc.

Major Cycle Computations

Explicit solutions of a variety of forms of the two-body equations can be found in the literature.[3, 5] One of the simplest solutions that is valid for circular and elliptical orbits is summarized below:

$$\begin{aligned} r^2 &= x^2 + y^2 + z^2 & s^2 &= \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \\ r\dot{r} &= x\dot{x} + y\dot{y} + z\dot{z} & a^{-1} &= 2r^{-1} - s^2\mu^{-1} \\ eCE &= 1 - ra^{-1} & eSE &= r\dot{r}(\mu a)^{-1/2} \\ & & n = \mu^{1/2}a^{-3/2} \end{aligned} \tag{9}$$

$$n\Delta t = \Delta E + eSE(1 - C\Delta E) - eCE\,S\Delta E \tag{10}$$

$$\bar{P}(\Delta t) = a/r(C\Delta E - eCE)\bar{P}(0) + 1/n[S\Delta E(1 - eCE) + eSE(1 - C\Delta E)]\bar{V}(0) \tag{11}$$

where in the foregoing and following, S stands for sine and C for cosine; $\bar{P}(\Delta t)$ is the position at time Δt; μ is the gravitational constant; x, y, and z are the coordinates of the initial position $\bar{P}(0)$; $\dot{x}$, $\dot{y}$, and $\dot{z}$ are the coordinates of the initial velocity $\bar{V}(0)$; r is radial distance; s is velocity magnitude; a is semimajor axis; e is eccentricity; E is initial eccentric anomaly; and n is mean orbital frequency.

The computations are straightforward, except (10) must be solved for ΔE. This can be done in a variety of ways. Newton's method works quite well with an initial guess of $\Delta E'$, where

$$\Delta E' = n\Delta t + [eCESn\Delta t + eSE(Cn\Delta t - 1)](1 + q) \tag{12}$$

$$q = eCE\,Cn\Delta t - eSE\,Sn\Delta t \tag{13}$$

Equations (9–11) can be evaluated by computing two square roots, solving Eq. (10) for ΔE, and computing $\sin\Delta E$ and $\cos\Delta E$. Approximate techniques can be used to update one solution from the previous one.

For instance, square roots can be updated by

$$f = (F + F/f')/2 \tag{14}$$

where F is the square and f' is the initial estimate. Sines and cosines can be updated for small angular changes by

$$s(x + \Delta x) = Sx + Cx\Delta x - Sx\Delta x^2/2 \tag{15}$$

$$C(x + \Delta x) = Cx - Sx\Delta x - Cx\Delta x^2/2 \tag{16}$$

The solution to Eq. (11) can be updated by

$$\left.\begin{aligned} eS(E + \Delta E') &= eSE\,C\Delta E' + eCE\,S\Delta E' \\ eC(E + \Delta E') &= eCE\,C\Delta E' - eSE\,S\Delta E' \end{aligned}\right\} \tag{17}$$

$$\delta = n\Delta t - \Delta E' - eSE + eS(E + \Delta E') \tag{18}$$

$$\Delta E = \Delta E' + \frac{\delta}{1 - eC(E + \Delta E')}\left[1 - \frac{\delta eS(E + \Delta E')}{2[1 - eC(E + \Delta E')]^2}\right] \tag{19}$$

where $\Delta E'$ is the previous value of ΔE. This solution is correct through terms of the second order in δ.

The approximations in Eqs. (14, 17, and 19) are stable in the sense that they converge on the correct answer. However, updating by (15) and (16) is unstable and will cause the sine and cosine to drift away from their true values. These deviations can be corrected by occasionally recomputing $\sin\Delta E$ and $\cos\Delta E$ from ΔE.

A possible implementation of the major cycle computations will now be described. $\bar{P}$ and $\bar{V}_R'$ are used to compute M and then to correct $\bar{V}_g$. Next, one component of the position, or of the estimated required velocity, is perturbed in order to compute partial derivatives. After making another correction to $\bar{V}_g$, a second component is perturbed, and so on, until all of the partial derivatives have been computed. Then, after the next $\bar{V}_g$ correction, the partial derivative matrix $[\partial\bar{P}(T)/\partial\bar{V}_R]$ is inverted, and the matrix $[\partial\bar{V}_R/\partial\bar{P}]$ is computed. After seven corrections to $\bar{V}_g$, all partial derivative matrices have been updated. The process continues until engine cutoff.

Results

The mechanization previously mentioned has been tested by simulating the conditions of lunar rendezvous after ascent from the surface of the moon using a 90° transfer orbit. The simulation included the steering equations and flight controls dynamics, and was intended to approximate the actual behavior of the vehicle. The results are shown in Table 1.

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Transient Surface Temperatures in Rocket Nozzles

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Nomenclature

c = specific heat of solid, Btu/lb-°F
d = half-thickness of node, in.
erf = error function
h = heat-transfer coefficient, convective plus radiant heating, Btu/sec-in.^{2}-°F
k = thermal conductivity of solid, Btu/sec-in.^{2}-°F/in.
M_n = eigenvalues of $M_n \tan M_n = N_{Bi}$
N_{Bi} = Biot number, $N_{Bi} = h\delta/k$
N_{Fo} = Fourier modulus, $N_{Fo} = \alpha\theta/\delta^2$
n = number of eigenvalues
T = temperature, °F
α = thermal diffusivity, in.^{2}/sec
δ = thickness of plate, in.
θ = time, sec
ρ = density, lb/in.^{3}

Subscripts

e = equivalent of solid material
g = gas
i = initial value at $\theta = 0$
w = surface of wall
r = recovery temperature

Basic study of heat flux, stress, chemical reaction and physical action at the inside surface of a rocket nozzle wall requires a knowledge of the wall surface temperature. To calculate the transient temperature response of the inside surface of the rocket nozzle, one can use either a simple analytical solution limited to the flat plate geometry,[1] or a digital computer solution based on finite difference techniques.[2] Efficient use of the first method depends upon knowing the number of eigenvalues necessary to obtain an accurate answer; this paper presents a correlation which will enable the user to predict an optimum number of eigenvalues at any specified time of heating desired. It will be shown that for more complex studies, and complex nozzle geometries, the second method is useful and is just as accurate as the first, provided that the depth of the first thickness increment (or "node") adjacent to heating surface is properly chosen.

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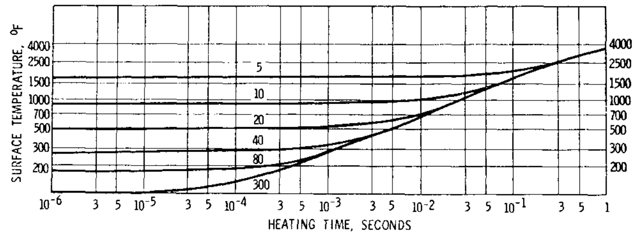

Fig. 1 Transient surface temperature for infinite flat plate calculated from Eq. (2) with varying number of eigenvalues.

Analytical Method (Flat Plate)

The equation for a semi-infinite solid,[1] with $\xi \equiv (\alpha\theta)^{1/2}h/k$, is

$$(T_w - T_i)/(T_r - T_i) = 1 - e^{\xi^2}[1 - \mathrm{erf}(\xi)] \qquad (1)$$

Equation (1) can be used only in flat-plate applications when the heating period (θ) is short, i.e., before the insulated back surface is heated. Typically, it is used for $\theta < 10^{-3}$ sec. For longer times, the equation for an infinite flat plate[1] should be used

$$\frac{T_w - T_r}{T_i - T_r} = 2 \sum_{n=1}^{n} \left(\frac{1}{1 + 2M_n/\sin 2M_n} \right) \exp(-M_n{}^2 N_{Fo}) \qquad (2)$$

In order to apply Eq. (2) correctly, a sufficient number of eigenvalues (which may become excessive for $\theta < 10^{-3}$ sec) must be used. For example, let $T_r = 6500$, $h = 0.03$, $k = 7 \times 10^{-4}$, $c = 0.3$, $\rho = 0.06$, $\delta = 1.5$, and $T_i = 70$°F (see nomenclature for units). For this case, Fig. 1 shows the T_w vs θ curves obtained with various numbers (n) of eigenvalues; for $\theta \geq 10^{-2}$ sec, $n = 40$ is sufficient, and for $\theta \geq 10^{-3}$, $n = 300$ is sufficient. The error for $\theta < 10^{-3}$ with $n = 300$ is seen, in Table 1, by comparing T_w's with those computed by Eq. (1).

To establish a method for determining the correct n to be used in Eq. (2), all parameters have to be generalized. For solid-propellant rockets, the common nozzle-insert materials are graphite (250°–7000°F) and molybdenum and tungsten (250°–5000°F).[3] Ranges of physical properties for these materials are:

$$3 \times 10^{-4} < k < 2 \times 10^{-3} \qquad 0.01 < c < 0.76$$
$$0.06 < \rho < 0.6 \qquad 10^4 < \alpha < 3 \times 10^5$$

Typical operating ranges for nozzle inserts are:

$$0.01 < h < 0.04 \qquad 0.1 < (T_w - T_r)/(T_i - T_r) < 1.0$$
$$0.125 < N_{Bi} < 33 \qquad 0.2 < N_{Fo} < 6.0$$

wherein the range of δ is assumed to be 0.3–3 in.

It has been found that practical correlations of the data are obtained by plotting θ vs $N_{Bi}\theta/N_{Fo}$ for various values of n, as in Fig. 2. The curves have been drawn through maximum time points to be conservative in estimating the minimum n for use in Eq. (2).

By appropriate use of Eqs. (1) and (2) as outlined previously, correct values for surface temperature can be obtained for any time period. These values then can be used to check surface temperatures obtained by a numerical method.

Numerical Method (Finite Differences)

Numerical methods (on computers) have obvious advantages, but their application to surface temperature calcu-

Table 1 Transient surface temperatures calculated by exact methods

	θ, sec	10^{-6}	10^{-5}	10^{-4}	10^{-3}	10^{-2}	10^{-1}
T_w, °F	Eq. (1)	76.0	90.0	130.0	256.8	638.6	1610.6
	Eq. $(2)_{n=300}$	100.0	103.7	132.5	257.2	638.7	1610.6